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# Introduction to Special Issue: Reconsidering Frege's Conception of Number

Dedicated to the memory of
Aldo Antonelli (1962/2/10–2015/10/11)

Before launching into the introduction to this issue, we would first like to mention a conclusion of sorts. The end in question is of many friendships; many productive collaborations; many days spent eating good food, drinking good wine (or beer), and talking great philosophy; and an end to a great many other wonderful things. Aldo Antonelli passed away, tragically and too young, on October 11 of 2015, when this special issue was near the end of the long path from initial ideas to eventual publication. We could easily justify dedicating this issue to Aldo based on the role he played in its production. He was involved in various ways from beginning to end, and there is no doubt that the issue would be less 'special' had he not added his insights and intelligence, but most importantly his generosity, to the project at various critical points. But that is not the only reason for the dedication. The editors of, and contributors to, this special issue all have fond memories of long conversations with Aldo — often about the very topics discussed in the essays collected here. We are deeply saddened by the fact that we will not have any more opportunities to talk with him about Frege, logicism, and many other things. Thus, we are not dedicating this issue to Aldo just because he was a good philosopher, but also because he was a good friend.

## 1. INTRODUCTION TO THE INTRODUCTION

Gottlob Frege's main project was to clarify the content and the status of mathematics. More specifically, he intended to show that arithmetic and analysis are simply a part of logic, thus securing the special epistemic status of this subject matter in the face of various empiricist and Kantian challenges. A central part of this logicist project was to analyze, or to reconstruct, the concept of

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number (understood in a broad sense) within logic, and he did so in several stages [Reck, 2005]. First, he developed the formal logical tools necessary for the project in *Concept Script* (*Begriffsschrift*, 1879). Then, in *The Foundations of Arithmetic* (*Die Grundlagen der Arithmetik*, 1884), he sketched a reconstruction of the cardinal numbers. In *Basic Laws of Arithmetic, Part I* (*Grundgesetze der Arithmetik*, *Band I*, 1893), he formalized and refined this treatment of the cardinal numbers, and finally, in *Basic Laws, Part II* (*Grundgesetze*, *Band II*, 1903), he began to extend his approach to real (and complex) analysis. It was only late in his life that Frege gave up on such aspirations.

Frege's logicist project led to many celebrated contributions that outlasted his logicism. These include but are not limited to: the development of modern relational and quantificational logic; pointed criticisms of other positions in the philosophy of mathematics, especially formalism and psychologism; and the distinctions between sense and reference and between concept and object. While certainly not uncontroversial, all of these contributions remain prominent and influential today. In contrast, Frege's reconstruction of the cardinal numbers and real numbers, based on a systematic theory of extensions (*i.e.*, classes), was seen as a failure from early on and, consequently, put aside for many years.

What seemed to doom Frege's original reconstruction, thus leading to its long-standing neglect, was Russell's antinomy. Russell informed Frege of the paradox in a famous letter from 1902, and, after a brief attempt to salvage a consistent version of logicism, Frege abandoned his version of the view. As a result, it was alternative forms of logicism, such as that presented in Whitehead and Russell's *Principia Mathematica* (1910–13), the formal-axiomatic approach championed by Hilbert (building on Peano and Dedekind), and the set-theoretic perspective codified in the Zermelo-Fraenkel axioms (including von Neumann's set-theoretic representations of cardinal and ordinal numbers) that received almost all the philosophical and mathematical attention. More recently, we can add category-theoretic approaches to this list.

The conviction that underlies the present special issue of *Philosophia Mathematica* is that it is nevertheless worthwhile, as well as timely, to reconsider Frege's philosophy of mathematics, and more specifically, his original conception of numbers. There are three main considerations behind this conviction:

First, the last 25–30 years have seen a new account of the philosophy and foundations of mathematics — neo-logicism — slowly gain philosophical prominence and technical sophistication. Neo-logicism, defended by Crispin Wright, Bob Hale, and their collaborators (including one of the editors of this special issue), builds self-consciously on Frege's approach. One point that the work of these neo-logicists (and their 'friendly critics', *e.g.*, George Boolos, John Burgess, William Demopoulos, Kit Fine, Richard Heck, Stewart Shapiro, and Alan Weir) has made evident is that it is not Frege's reconstruction of the cardinal numbers that falls prey to Russell's antinomy, at least not directly, but rather Frege's theory of extensions (and value ranges more generally). Nevertheless, the precise relationship between Frege's original conception of number and Russell's antinomy remains only partially understood. One goal of the present issue is to call further attention to this issue and begin to remedy this oversight.

A second reason for reconsidering Frege's conception of number is that new research on the early history of analytic philosophy, on the one hand, and on the history of the philosophy of mathematics and logic, on the other, has led to a much fuller, more sophisticated understanding of the significance of and motivations behind Frege's work, and of its connections to other philosophical and mathematical research conducted in the late nineteenth and early twentieth centuries. A number of essays in this special issue concern, either directly or indirectly, these more historically nuanced perspectives on Frege's work, and thus contribute directly to this recent, more historically sensitive, approach to philosophy of mathematics during this period.

Third, the text Frege considered his *magnum opus* — *Basic Laws of Arithmetic, Vols. I–II* — has only recently been published in a full English translation [Ebert and Rossberg, 2013]. Having this new edition of *Basic Laws* available will surely stimulate further work on Frege's philosophy of mathematics, and suggest new interpretations and novel avenues of investigation within Frege studies. In fact, this 'new wave' of research on Frege's logic, mathematics, and philosophy has already begun, as can be seen by the direct and substantial influence of the new translation on the essays in this special issue. (For more, compare [Ebert and Rossberg, forthcoming].)

For present purposes, it will be helpful to distinguish three distinct but interconnected aspects of Frege's original reconstruction of the concept of number:

1. The logic Frege introduced as the systematic framework within which logicism was to be developed, and especially the theory of extensions (*i.e.*, classes) and value ranges included in this formalism.
2. Frege's definitions of the cardinal numbers and real numbers as equivalence classes, including both their precise mathematical construction and their philosophical motivation.
3. Frege's logicist treatment of mathematical induction and related issues, based on his new logic and his logical analysis of the ancestral relation.

In Frege's own writings, these three sides appear to be treated in roughly reverse order: (3) is first addressed in *Concept Script* (1879); (2) is addressed in *Foundations* (1884); and (1) is finally developed in full in *Basic Laws, Vol. I* (1893). While this impression is correct, loosely speaking, it does obscure some complexities in the development of Frege's thought, since much of Frege's novel logical apparatus other than extensions and value ranges was introduced already in *Concept Script*, and Frege's treatments of both the ancestral, in *Concept Script*, and his definition of cardinal numbers, in *Foundations*, are revised in various ways in *Basic Laws*.

In this special issue, we will provide new perspectives on each of these three aspects of Frege's project, including making more evident some of the complexities just mentioned. In doing so, the discussions will focus on the case of the cardinal numbers, as many of the core questions and problems already arise for them (and, at any rate, Frege never completed his account of the real and complex numbers).

## 2. FREGE'S LOGIC, BASIC LAW V, AND RUSSELL'S ANTINOMY

One of Frege's lasting contributions to both mathematics and philosophy is his development of many of the familiar tools of modern logic in the form of an idiosyncratic version of higher-order logic together with a theory of extensions. He not only introduced both of these explicitly and precisely, but also illustrated their power by making them central ingredients in his logicist treatment of what he took to be the central problems in the philosophy of mathematics.

As already mentioned, the form taken by Frege's logic changed over time. Initially, in *Concept Script*, he presented a version of higher-order logic, or simple type theory, that did not yet deal with extensions. In *Foundations*, he started to appeal to extensions of concepts in his informal reconstruction of the natural numbers. But it was only in *Basic Laws, Vol. I* that he added a formal systematic theory of extensions in terms of his now infamous Basic Law V. Actually, what he really added in *Basic Laws* was a theory of value ranges for functions generally, with extensions of concepts as a special case (since for Frege concepts are just functions from the domain to the set of the truth values {T, F}). In his later work, after having been confronted with Russell's antinomy, he eliminated the problematic theory of value ranges, retaining only his higher-order inferential framework [Reck and Awodey, 2004].

There are various questions about Frege's Basic Law V that have played a prominent role in the secondary literature. One strand involves asking how Frege could have thought of it as a logical law even before he found out about Russell's antinomy. This issue has gained renewed urgency in recent years because of parallel questions about the logical and epistemic status of Hume's Principle (or better, the Cantor-Frege Principle) and similar abstraction principles appealed to in neo-logicism. A prominent attempt to defend neo-logicism, as both a form of logicism and as a viable account of the nature of mathematics more generally, involves the claim that the two sides of, say, the bi-conditional that constitutes Hume's Principle express the same sense while 'carving up' that sense in different ways. Along similar lines, some scholars have suggested that Frege might have assumed that the two sides of Basic Law V involve this kind of 'content re-carving', which would then explain its status as a logical law.

In his contribution to the present special issue — 'Frege on sense identity, Basic Law V, and analysis' — Philip Ebert examines and rejects this suggestion, arguing that such a 'sense-identity assumption' is in conflict with other Fregean commitments. In doing so, he fleshes out a developmental story about Frege's thought that has consequences for his conceptions of logic and analysis as well. Ebert presents this explicitly as a contribution to our historical understanding of Frege and his work; but if correct, his argument also raises questions about certain aspects of neo-logicism, including whether it is 'Fregean' in this respect.

A second question about Frege's Basic Law V, again made prominent in connection with neo-logicism recently, is why he introduced this law, and the theory of extensions and value ranges that results, in the first place. As is well known by now, his *Foundations* and *Basic Laws* contain all the definitions and main proof steps needed for deriving arithmetic directly from Hume's Principle within second-order logic — a result that has come to be known as 'Frege's Theorem' [Heck, 2011]. Frege, however, was not content with this derivation, and there is a puzzle about why exactly.

In her article in this special issue — 'The breadth of the paradox' — Patricia Blanchette addresses this puzzle head on. She argues that Frege's strict requirements for existence proofs, which reveal a conception of models, consistency, and existence in mathematics that was quite different from later views (in Hilbert's wake), are crucial for understanding why Frege was unsatisfied with Frege's Theorem. Blanchette goes on to argue that Frege's views on these issues are also very different from neo-logicist views; and as a result, a neo-logicist-style response to Russell's antinomy (*i.e.*, reject Basic Law V in favor of Hume's Principle) is not an option for Frege. If correct, this explains why he was so devastated about the paradox — that is, why he came to think of it as undermining any possible logicist grounding of arithmetic, not just his own. And it again challenges the self-conception of neo-logicists to be carrying out a genuinely 'Fregean' project.

## 3. FREGE'S CONCEPTION, THE FREGE-RUSSELL NUMBERS, AND RUSSELL'S ANTINOMY

Frege's original conception of number, unlike that defended in neo-logicism, involves equivalence classes in a central and ineliminable way. This is true both for the cardinal numbers and the real numbers, although we will, again, focus on the former case here. In this context, a basic clarification is worth making right away. It is often assumed that Frege worked with what is often called 'the Frege-Russell conception of number', where the cardinal numbers are defined as equivalence classes (extensions) of equinumerous classes (extensions). But this is not true strictly speaking — neither for the early nor the later Frege. In *Foundations*, where he deals with this issue for the first time, Frege defines cardinal numbers as equivalence classes of equinumerous concepts. And when he updates that definition in *Basic Laws*, as part of his more mature and systematic theory, it is not by switching to equivalence classes of classes (*i.e.*, of extensions of concepts alone), but to equivalence classes of value ranges more generally. In other words, the items compared via the equinumerosity relation are now not only value ranges of concepts (*i.e.*, extensions), but also value ranges of functions that are not concepts (see [Cook, 2013]).

Nevertheless, for our purposes it helps to take another look at the traditional Frege-Russell conception and reconsider the background and motivation for Frege's approach in this connection. In particular, how is he led to using equivalence classes (of concepts, extensions, or value ranges) originally? And why does his use of them change as we pass from *Foundations* to *Basic Laws*?

A first point to note here is that Frege is trying to get a better grip on what cardinal numbers are in virtue of how they are used — that is, he is attending to the manner is which cardinal numbers are used to answer 'How many ( )?' questions. If we then ask what should figure in the empty slot ( ), it becomes apparent that it has to be something like concepts or their extensions. (Frege does not like to appeal to 'sets' here, since he associates that term with crude mereological views. Similiarly, we have avoided talk of 'classes' whenever possible — to avoid illicit identification of Frege's notion of extension with modern notions of set or class.)

Moreover, we get the same answer to 'How many $F$s?' and 'How many $G$'s?' exactly when the concepts $F$ and $G$, or the extensions determined by each of them, are equinumerous — that is, when the objects falling under them can be 1-1 correlated. Individual cardinal numbers correspond, thus, to equivalence classes of equinumerous concepts (or of equinumerous extensions of concepts). It is a small step, then, simply to identify the numbers with those equivalence classes, as Frege does. Why the additional switch to value ranges, as opposed to concepts or their extensions, in *Basic Laws*? The answer here has to do with pressures Frege felt concerning the systematic form of his logic: he thought of the notions of functions and their value ranges as having the right generality for his purposes, with concepts and their extensions being but a special case.

However, why return to examine Frege's definition of cardinal numbers as equivalence classes at all, given that Russell's antinomy undermines Frege's approach, as already discussed above? In particular, is there more than mere historical or antiquarian interest in reconsidering these definitions? An initial response, already hinted at above, is that Russell's antinomy directly undermines only Basic Law V, while the use of equivalence classes is affected at most indirectly. But more can and should be said than this. For example, it has been known for a while that the Frege-Russell construction is available in theories of classes/extensions other than Frege's (even if it is not available in ZFC, since the equivalence classes in question are 'too big' to count as sets). For example, Richard Heck [1996] shows that Frege's account of cardinal numbers as equivalence classes of extensions can be reconstructed consistently within predicative second-order logic.

Another prominent example of such a Frege-number-countenancing theory is Quine's New Foundations [Rosser, 2008]. It has also been known for a while that NF with urelements is consistent; moreover, recently progress has been made towards showing that NF itself is also consistent (relative to ZFC). But then, both the traditional Frege-Russell construction and the actual construction given by Frege in *Basic Laws* are separable from Russell's antinomy in a much deeper sense than previously acknowledged. As a result, explaining exactly why the construction is possible in NF — that is, explaining what prevents it from leading to a contradiction in this context — is of obvious importance for fully understanding why the antinomy does arise in other, more familiar set- or extension-theoretic contexts. This question is the topic of Thomas Forster's article in this special issue, called 'Mathematical objects arising from equivalence relations and their implementation in Quine's NF'.

If NF is, indeed, relatively consistent, this rehabilitates both the traditional Frege-Russell conception and Frege's variant of it to some degree. But does it rehabilitate any of them as a logicist or neo-logicist option? One way to approach that question is explored in Roy Cook's contribution to this special issue, 'Frege's cardinals and neo-logicism'. Cook asks whether the following three things can be combined consistently: (i) the adoption of the neo-logicist framework, where mathematical objects are introduced via abstraction principles similar to Hume's Principle; (ii) the Frege-Russell construction of cardinal numbers as equivalence classes of classes or extensions; and (iii) a defense of the claim that the abstraction principle governing such classes or extensions is logical in the relevant sense. Cook adopts a version of the Tarskian invariance conception of logic, as prominently discussed in recent philosophy of mathematics, in explicating the third criterion, and he arrives at a negative answer to this question — he proves that these three desiderata, on his preferred understanding of them, are incompatible. This is not necessarily the end of the story, however, as he concludes the paper with a discussion of alternative, perhaps less demanding, construals of these three constraints.

## 4. FREGE'S ORIGINAL TREATMENT OF MATHEMATICAL INDUCTION

The fifth article in this special issue, by Richard Heck, is titled 'Is Frege's definition of the ancestral adequate?' Unlike the other contributions, Heck's paper is motivated neither by issues arising out of neo-logicism nor by determining whether Frege's equivalence-class approach to numbers can be saved. Instead, it provides a good example of new work based on carefully reconsidering Frege's *Basic Laws*, focused on re-analyzing heretofore neglected parts of it [Heck, 2013].

The core question in Heck's article is whether Frege's definition of the ancestral relation in *Basic Laws*, and thus his treatment of mathematical induction, is circular, as some of Frege's contemporaries claimed, including Bruno Kerry and, more prominently, Henri Poincaré. After carefully reconstructing the relevant parts of the text, Heck argues that a slight modification of Frege's procedure allows for a compelling answer to the circularity objection. If correct, this rehabilitates yet another aspect of Frege's approach to the foundations of mathematics. Furthermore, Heck's careful consideration of early criticisms of Frege's (and Russell's) logicism by Poincaré, Kerry, and their contemporaries also constitutes a significant contribution to the history of the philosophy of mathematics and logic during that period.

A final remark concerning the genesis of the articles in this special issue should be added. The majority of them were presented, in earlier versions, in a workshop at the Munich Center for Mathematical Philosophy in January of 2013, while one of us (Erich Reck) was a research fellow there. We would like to thank the directors of the MCMP, especially Hannes Leitgeb, for allowing us to organize that workshop and for supporting it both logistically and financially.

We could not have asked for a more congenial place to think about Frege's conception of number.

Erich H. Reck
(Department of Philosophy, University of California, Riverside)
Roy T. Cook
(Department of Philosophy, University of Minnesota – Twin Cities)
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